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INVARIANTIVE CHARACTERIZATION OF SOME LINEAR ASSOCIATIVE ALGEBRAS.

BY OLIVE C. HAZLETT.*

Linear associative algebras of a small number of units, with coördinates in the field of ordinary complex numbers, have been completely tabulated,† and their multiplication tables have been reduced to very simple forms. But if we had before us a linear associative algebra, the chances are that its multiplication table would not be in such a form that we could find out readily to what standard form it was equivalent. And so the question arises, "May we not find invariants which completely characterize these algebras?"

We use the term invariant here in the sense defined by Professor Dickson,—that is, a single-valued function (in Dirichlet's sense) of the constants of multiplication which takes on the same value for two algebras of the same number of units (having constants of multiplication and coördinates in any field F) if the two algebras be equivalent with respect to F ,—that is, if there be a non-singular linear homogeneous transformation‡ with coefficients in F which will transform the units of one algebra into the units of the second algebra. The invariants $I_1, \dots, I_m$ are said to completely characterize the algebras of n units over a field F when each I_k has the same value for two algebras only when they are equivalent.

It turns out that, in the field C of ordinary complex numbers, under the general group T of linear transformations, the binary and ternary linear associative algebras with a modulus are both completely characterized by the rank of the algebra and by the rank of certain homogeneous quadratics in the constants of multiplication and the coördinates of the general number of the algebra.

In order to find invariants in the more technical sense defined above, we will first find some rational functions which, in the more usual sense, remain invariant under any linear transformation on the units. We will mean by a *rational invariant* of the n -ary linear algebra $(e_1, \dots, e_n)$ with

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† See especially B. Peirce: "Linear Associative Algebras," Am. Jour. of Math., vol. 4 (1881), p. 97-192; G. Scheffers: "Zurückführung complexer Zahlensysteme auf typische Formen," Math. Ann., vol. 39 (1891) p. 293-390; Study: "Complexen Zahlen," Gött. Nach., 1889, p. 237-268.

‡ Hereafter in this paper, the word transformation will be used to denote a non-singular linear homogeneous transformation.

coördinates in F and indeterminate constants of multiplication γ_{ijk} ($i, j, k = 1, \dots, n$), a rational function of the γ 's which, when the units of the algebra are subjected to any linear transformation with coefficients in F , is merely multiplied by some power of the determinant of the transformation. *Rational covariants* are defined in a similar manner.

1. Rational Invariants of General Linear Algebras.—Both characteristic determinants*

$$\delta \equiv \left| \sum_i \gamma_{ijk} x_i - d_{jk} \omega \right|, \quad \delta' \equiv \left| \sum_i \gamma_{jik} x_i - d_{jk} \omega \right| \quad (j, k = 1, \dots, n)$$

are rational covariants and thus produce invariants. For the right-hand determinant δ is the determinant of the n linear homogeneous equations

$$\sum_{i,j=1}^n (\gamma_{ijk} x_i - d_{jk} \omega) y_j = 0 \quad (k = 1, \dots, n),$$

which are the necessary and sufficient conditions that it be possible to find a scalar ω and a number $Y = \sum_i y_i e_i$ such that $XY = \omega Y$.† But a necessary and sufficient condition that there be a solution $Y \neq 0$ is that ω be a root of $\delta = 0$. Suppose ω_0 is such a root, or $XY = \omega_0 Y$. If the units of the algebra be subjected to a non-singular linear transformation (with coefficients in F) such that x_i goes into X_i and γ_{ijk} in Γ_{ijk} , the two quantities X and Y are unchanged and thus ω_0 is unchanged. But ω_0 must be a root of $\delta(X_i, \Gamma_{ijk})$ by the foregoing, and hence $\delta(X_i, \Gamma_{ijk})$ vanishes with $\delta(x_i, \gamma_{ijk})$.

Moreover δ is an irreducible function of ω where the constants of multiplication are indeterminates. Suppose $\delta = \delta_1 \delta_2$. No one γ_{ijk} can occur in both δ_1 and δ_2 ; and moreover if δ be actually reducible δ_1 and δ_2 must each contain γ 's of the form γ_{ijj} . Suppose γ_{i11} be in δ_1 ; then γ_{j1k} and γ_{jk1} cannot occur in δ_2 ($j, k = 1, \dots, n$). Similarly, if γ_{i1l} be one of the γ 's which actually occur in δ_2 , γ_{jlk} and γ_{jkl} ($j, k = 1, \dots, n$) cannot occur in δ_1 . Or γ_{j1l} and γ_{j1l} ($j = 1, \dots, n$) cannot occur in δ_1 or δ_2 , which is impossible.

But, if a polynomial φ be irreducible and if a polynomial ψ vanish whenever φ vanishes, then φ is a factor of ψ . Consequently, since the coefficient of ω^n is $(-1)^n$ in δ formed for both the original and transformed algebra, δ is an absolute covariant. Similarly with δ' . Furthermore, since ω is an arbitrary variable, the coefficients of the powers of ω in δ and δ' must separately be absolute covariants.

If we write $\delta = \sum_{i=0}^n C_i \omega^i$ and $\delta' = \sum_{i=0}^n C'_i \omega^i$, then the C 's and C' 's are homogeneous polynomials in the coördinates, with coefficients homogeneous in the γ 's. Now these polynomials produce invariants of the algebra.

* ω is an arbitrary variable which may take on values in field F or in the algebra; and the n x_i 's are the coördinates of any number in the algebra.

† Cartan: *Annales de Toulouse*, séries I, vol. 12 (1898); B. 16.

We will mean by a rational covariant of a covariant K of the n -ary linear algebras with coördinates in the field F and undetermined γ 's, a rational function K' of the coördinates and of the coefficients of K which, when the coördinates are subjected to a linear transformation (with coefficients in F), is merely multiplied by some power of the determinant of the transformation. If K' do not contain the coördinates, it will be called a rational invariant of K . Just as, in the invariant theory for quantics, any covariant or invariant of a covariant of a quantic is itself a covariant or invariant of the quantic, so any covariant or invariant of a covariant of a linear algebra is itself a covariant or invariant of the algebra. Hence we have

THEOREM I. *The rational, integral invariants and covariants of the characteristic determinants and of the coefficients of the powers of ω therein are respectively invariants and covariants of the n -ary linear algebras with coördinates and constants of multiplication in any field.*

2. Ternary Linear Associative Algebras with a Modulus.—We are now in a position, from Study's results,* to exhibit some invariants which completely characterize linear associative algebras of two and three units having a modulus.† In this article, Study separates into types all linear associative algebras of two, three and four units having a modulus, according to the rank and according to the character of the rank equation as shown by its number of multiple roots and their multiplicities. He takes as his field F the field C of all complex numbers. His results for $n = 3$ follow.

$$r = 3\ddagger$$

$$\text{I} \quad \begin{array}{c} e_1 \quad e_2 \quad e_3 \\ e_1 \left| \begin{array}{ccc} e_1 & 0 & 0 \\ e_2 & 0 & e_2 & 0 \\ e_3 & 0 & 0 & e_3 \end{array} \right. \end{array}$$

$$\text{II} \quad \begin{array}{c} e_1 \quad e_2 \quad e_3 \\ e_1 \left| \begin{array}{ccc} e_1 & e_2 & 0 \\ e_2 & e_2 & 0 & 0 \\ e_3 & 0 & 0 & e_3 \end{array} \right. \end{array}$$

$$\text{III} \quad \begin{array}{c} e_1 \quad e_2 \quad e_3 \\ e_1 \left| \begin{array}{ccc} e_1 & e_2 & e_3 \\ e_2 & e_2 & e_3 & 0 \\ e_3 & e_3 & 0 & 0 \end{array} \right. \end{array}$$

$$\delta = \delta' = \prod_{i=1}^3 (x_i - \omega), \quad \delta = \delta' = (x_1 - \omega)^2(x_3 - \omega), \quad \delta = \delta' = (x_1 - \omega)^3.$$

* See note 1, page 1.

† That is, a number E having the properties of ordinary unity.

‡ $r = 3$ means that three is the degree of the equation of lowest degree satisfied by every number in the algebra; or that the rank is three. $r = 2$ means the rank is two. Clearly the rank is an invariant in the more technical sense.

$$r = 2^*$$

IV

$$\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & e_1 & e_3 \\ e_3 & e_3 & -e_3 & 0 \end{array}$$

V

$$\begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ e_1 & e_1 & e_2 & e_3 \\ e_2 & e_2 & 0 & 0 \\ e_3 & e_3 & 0 & 0 \end{array}$$

$$\delta = (x_1 + x_2 - \omega)(\omega^2 - 2x_1\omega + x_1^2 - x_2^2), \quad \delta = \delta' = (x_1 - \omega)^3.$$

$$\delta' = (x_1 - x_2 - \omega)(\omega^2 - 2x_1\omega + x_1^2 - x_2^2),$$

Since the rank of a quadratic form is invariant, and since for two algebras of the same rank but of different types, the rank of C_1 is different, we see that

THEOREM II. *In the field of all complex numbers the ternary linear associative algebras with a modulus are completely characterized by their rank and by the rank of the coefficient of ω in either characteristic determinant.*

Type	Rank	Rank of C_1
I	3	3
II	3	2
III	3	1
IV	2	2
V	2	1

3. General Binary Linear Associative Algebras.—The invariants which characterize the binary linear associative algebras with a modulus can be obtained in a similar manner from Study's results for the binary algebras; but we can do better—by reduction from the ternary algebras with a modulus, we can find the invariants which characterize all binary linear associative algebras (with or without a modulus).

An n -ary linear algebra $(e_1, \dots, e_n)$ with coördinates and constants of multiplication in a field F is said to be *reduced* when in every number of this algebra we set $x_{m+1} = \dots = x_n = 0$. The system thus defined may or may not be closed,—i. e. the product of any two numbers in the system may or may not be in the system. If such products always remain in the system, and if the coördinates of $(e_1, \dots, e_m)$ range over F , the numbers are said to form a *subalgebra* of the original algebra.

In particular, take the ternary linear algebras (not necessarily associative) over a field F where e_3 is the modulus, and reduce by setting $x_3 = 0$;

* Study says that, in this case "we will choose two units beside $e_1 = 1$ such that either $e_2^2 = e_3^2 = e_1$ or $e_2^2 = e_3^2 = 0$ "; but conceivably there might be an algebra in which we could not do this,—where at best we could choose units such that $e_2^2 = e_1, e_3^2 = 0$. Such however is readily shown impossible for $r = 2$.

and consider the resulting binary algebras. For convenience, two right-hand (or two left-hand) characteristic determinants for binary algebras over a field F will be said to be of the same type if the discriminants of their constant terms be the same except for a non-zero factor which is a perfect square in F ; and if, when the coefficient of ω vanishes identically in one determinant, it does in the other also. Since the invariants which characterize the initial algebra necessarily characterize the sub-algebras, no two non-equivalent ternary linear algebras can, by this reduction, produce two equivalent binary algebras. Finally, since by adjoining a modulus to a binary algebra, we have a ternary algebra with a modulus, these are all the binary linear algebras. Thus the binary algebras are partially characterized by the type of their characteristic determinants and by any set of invariants which completely characterize the algebras with three units. But how far do these characterize them? how many non-equivalent binary algebras are produced by any one type of ternary algebras?

If two subalgebras $a : (e_1, e_2)$ and $a' : (e_1', e_2')$ obtained by this reduction from equivalent ternary algebras are not equivalent, then one or both units of a' are linearly independent of the units of a . In the second case, since there are only three linearly independent numbers in the original algebra, the units of a' can be so transformed that we have the first case.

Now this first case is possible only if the two algebras can be transformed such that their units are respectively of the forms (e_1, e_2) and $(e_1' = E - e_1, e_2)$ where E is the modulus of the original algebra. For E must be expressible as the sum of two numbers, one in a , the other in a' . Suppose $e_2' = b_1e_1 + b_2e_2$, $E = \epsilon_1e_1 + \epsilon_2e_2 + \epsilon_1'e_1' + \epsilon_2'e_2'$. If in a , we set $E_1 = \epsilon_1e_1 + \epsilon_2e_2$, $E_2 = b_1e_1 + b_2e_2$, the transformation is non-singular. For otherwise E_1 would be a scalar multiple of e_2' and thence E would be in a' , which is impossible. Similarly, since $E - E_1$ is linearly independent of e_2' , we can set $E_1' = E - E_1$ and $E_2' = e_2'$ and the algebras are in desired form.

From the identity $e_1'e_2' = e_2 - e_1e_2$ we infer that $e_1e_2 = me_2$; and similar arguments for the other products show that the multiplication tables for these two algebras are of the forms

$$\begin{array}{cc}
 a: & \begin{array}{cc} e_1 & e_2 \\ e_1 \left| \begin{array}{cc} e_1 + he_2 & me_2 \\ e_2 \left| \begin{array}{cc} ne_2 & le_2 \end{array} \end{array} \right. & \\
 a': & \begin{array}{cc} e_1' & e_2' \\ e_1' \left| \begin{array}{cc} e_1' + he_2' & (1-m)e_2' \\ e_2' \left| \begin{array}{cc} (1-n)e_2' & le_2' \end{array} \end{array} \right. &
 \end{array}
 \end{array}$$

Thus there are at most two non-equivalent binary algebras produced from a ternary algebra by the above reduction. Moreover the right-hand characteristic determinants for these algebras are not of the same type, for they

are

$$a: \quad \delta = \omega^2 - \omega[(m+1)x_1 + lx_2] + x_1[mx_1 + lx_2],$$

$$a': \quad \delta = \omega^2 - \omega[(2-m)x_1 + lx_2] + x_1[(1-m)x_1 + lx_2].$$

Hence, by theorem I and the argument at the beginning of this section, we have

THEOREM III. *The binary linear algebras over any field F are characterized completely by the type of their characteristic determinants and by any set of invariants which completely characterizes the ternary algebras with a modulus.*

The binary linear associative algebras over the field of all real and complex numbers are as follows:

I	II	III	IV
$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & e_1 & 0 \\ e_2 & 0 & e_2 \end{array}$	$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & e_1 & e_2 \\ e_2 & e_2 & 0 \end{array}$	$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & e_1 & 0 \\ e_2 & 0 & 0 \end{array}$	$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & e_2 & 0 \\ e_2 & 0 & 0 \end{array}$
$\delta = \delta' = (x_1 - \omega)(x_2 - \omega),$	$\delta = \delta' = (x_1 - \omega)^2,$	$\delta = \delta' = -\omega(x_1 - \omega),$	$\delta = \delta' = \omega^2,$

V	VI	VII
$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & 0 & e_1 \\ e_2 & 0 & e_2 \end{array}$	$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & e_1 & e_2 \\ e_2 & 0 & 0 \end{array}$	$\begin{array}{c cc} & e_1 & e_2 \\ e_1 & 0 & 0 \\ e_2 & 0 & 0 \end{array}$
$\delta = -\omega(x_2 - \omega),$	$\delta = (x_1 - \omega)^2,$	$\delta = \delta' = \omega^2.$
$\delta' = (x_2 - \omega)^2,$	$\delta' = -\omega(x_1 - \omega),$	

Of these, I, IV and VII are obtained by reduction from types I, III and V respectively in Study's list of ternary algebras; whereas II and III are given by type II of these ternary algebras, and V and VI by type IV.* All algebras but the last are completely characterized by the type of the characteristic determinants; and thus we have

THEOREM IV. *The non-trivial linear associative binary algebras are completely characterized by the rank of C_0 and C'_0 , and by the identical vanishing or non-vanishing of C_1 .*

* Peirce (loc. cit., pp. 120-122) says there are only three irreducible (or pure) linear associative algebras of two units and one irreducible (or mixed) algebra. The three algebras he tabulates are of types II, IV and VI in the above list. Cayley, I find, points out this error in "Double Algebra," Proceedings London Math. Soc., vol. XV (1884); Collected Works, XII, p. 60. He here gives all the seven types of binary associative algebras. Like Peirce, he gets them directly and not by reduction.